## 4. Generating functions

**To read:**
[3] Chapters 12.1, 12.2.

### 4.1. Combinatorial applications of polynomials.

*Example.* How many ways are there to pay the amount of 21 francs with 6 one-francs coins, 5 two-francs coins, and 4 five-francs coins? The requited number is in fact the number of solutions of the equation

$$x_1 + x_2 + x_3 = 21, \tag{2}$$

with $x_1 \in \{0, 1, 2, 3, 4, 5, 6\}, x_2 \in \{0, 2, 4, 6, 10\}$, and $x_3 \in \{0, 5, 10, 15, 20\}$. In order to compute this, we associate to each variable $x_i$ a polynomial $p_i$ as follows:

$$\begin{aligned} p_1(x) &= 1 + x + x^2 + x^3 + x^4 + x^5 + x^6, \\ p_2(x) &= 1 + x^2 + x^4 + x^6 + x^8 + x^{10}, \\ p_3(x) &= 1 + x^5 + x^{10} + x^{15} + x^{20}. \end{aligned}$$

The number of solutions of equation (2) above will be the cofficient of $x^{21}$ in the product $p_1(x)p_2(x)p_3(x)$.

**Exercise 1.** A box contains 30 red, 40 blue, and 50 white balls; balls of the same color are indistinguishable. How many ways are there of selecting a collection of 70 balls from the box?

### 4.2. Multinomial theorem.

**Theorem 4.1.** *(Multinomial theorem). The following holds:*

$$(x_1 + x_2 + \ldots + x_n)^k = \sum_{\substack{i_1, i_2, \ldots, i_n \geq 0 \\ i_1 + i_2 + \ldots + i_n = k}} \frac{k!}{i_1!\, i_2! \cdots i_n!} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}.$$

### 4.3. Calculation with power series.

**Definition 4.2.** Let $(a_0, a_1, \ldots)$ be a sequence of real numbers. Then, its *generating function* $a(x)$ is

$$a(x) = a_0 + a_1 x + a_2 x^2 + \ldots.$$

**Theorem 4.3.** *Let $a_0, a_1, \ldots$ be a sequence of real numbers. If $|a_k| \leq c^k$ for every $k$, where $c$ is a positive real constant, then the series*

$$a_0 + a_1 x + a_2 x^2 + \ldots$$

*is convergent for all $x$ with $|x| < \frac{1}{c}$.*

*Proof.* Since $|a_k| \leq c^k$ for ever $k$, we have

$$\sum_{k=0}^{\infty} \left| a_k x^k \right| = \sum_{k=0}^{\infty} |a_k|\, |x|^k \leq \sum_{k=0}^{\infty} |cx|^k.$$

Furthermore $|x| < \frac{1}{c}$, therefore $|cx| < 1$ for every $k$. Next we show $1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$ for $x \in (-1, 1)$: Let $s = 1 + x + x^2 + x^3 + \ldots + x^{n-1}$, then $xs = x + x^2 + x^3 + \ldots + r^n$ and therefore $s - xs = 1 - x^n$. Thus $s = \frac{1-x^n}{1-x}$ for $x \neq 1$. If $|x| < 1$ the series converges as $n$ goes to infinity. Therefore, we have

$$1 + x + x^2 + x^3 + \ldots = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \text{ for } |x| < 1.$$

Since $|cx| < 1$, we get $\sum_{k=0}^{\infty} |cx|^k = \frac{1}{1-|cx|}$. We have shown that $\sum_{k=0}^{\infty} a_k x^k$ is absolutely convergent, hence it is convergent. □

4.4. **Examples of generating functions.** Consider the following two examples.

**Example 1.** Consider the sequence $a_n = n + 1$, $n \in \mathbb{Z}_{\geq 0}$. Then the generating function is

$$A(x) = 1 + 2x + 3x^2 + \ldots = \frac{d}{dx}(1 + x + x^2 + \ldots) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

**Example 2.** Consider the sequence $b_n = (n+1)^2$, $n \in \mathbb{Z}_{\geq 0}$. Arguing in a similar way, one gets that the generating function is $B(x) = \frac{d}{dx}A(x) - A(x)$.

**Exercise 2.** What is the generating function of the sequence $(a_0, a_1, \ldots)$ with $a_k = 2^{\lfloor k/2 \rfloor}$?

**Theorem 4.4.** *(Generalized binomial theorem). For every $r \in \mathbb{R}$ and every integer $n \geq 0$, let*

$$\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}.$$

*Then, the following holds:*

$$(1+x)^r = \binom{r}{0} + \binom{r}{1}x + \binom{r}{2}x^2 + \cdots$$

*for every $x$ with $|x| < 1$.*

*Proof.* Let $f(x) = (1+x)^r$, then $f^{(n)}(0) = r(r-1)(r-2)\cdots(r-n+1)$. Since $\binom{r}{n} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}$, we have $\binom{r}{n} = \frac{f^{(n)}(0)}{n!}$. For a series $a(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ the element $a_n$ is uniquely determined by $a_n = \frac{a^n(0)}{n!}$. Therefore $(1+x)^r = \binom{r}{0} + \binom{r}{1}x + \binom{r}{2}x^2 + \ldots + \binom{r}{n}x^n + \ldots$.

Next we have to show that the series converges for $|x| < 1$: The series $\sum_{n=0}^{\infty} \binom{r}{n} x^n$ converges if

$$\lim_{n\to\infty} \left| \frac{\binom{r}{n+1} x^{n+1}}{\binom{r}{n} x^n} \right| < 1.$$

This is the case if

$$\lim_{n\to\infty} \left| \frac{n+1}{n-r} x \right| < 1.$$

which holds for $|x| < 1$. □

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## References

[1] Discrete Mathematics (L. Lovasz, J. Pelikan , K. Vesztergombi);
[2] Combinatorics: Set Systems, Hypergraphs, Families of Vectors and Combinatorial Probability (B. Bollobas);
[3] Invitation to Discrete Mathematics (J. Matousek, J. Nesetril).
[4] Extremal combinatorics (S. Jukna).
[5] Combinatorial theory (M. Hall), Blaisdell publishing company, 1967.